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Condition for Injectivity of Global Maps for Tessellation Automata

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In this paper, we introduce two notions of being "balanced" and being "hard." And we prove that these notions are necessary and sufficient conditions for global maps $\bar{C} \to \bar{C}$ and $C \to C$ of tessellation automata to be one to one, respectively, where $C(\bar{C})$ denotes the set of all the (finite) configurations of tessellation automata.

Patt (1972) proved that if the global map defined from a local map of a one-dimensional tessellation automaton (TA) is one to one, then the local map is necessarily "balanced." We introduce a new notion of being "balanced" which is stronger than Patt's. Then we prove that a global map of TA with an arbitrary dimension has this property if and only if the global map τ: $\bar{C} \to \bar{C}$ is one to one, where $\bar{C}$ denotes the set of all the finite configurations for the TA. Next we define the notion of being "hard" that is stronger than that of being "balanced." And we prove a global map of TA with an arbitrary dimension has this property if and only if the global map τ: $C \to C$ is one-to-one, where C denotes the set of all the configurations for the TA, including infinite configurations.

Let $\Sigma = \{0, 1,..., q-1\}$. Σ represents the set of states that can be assumed by each machine of TA. Let Z denote the set of all integers. In this paper, we are concerned exclusively with two-dimensional TA without loss of generality. Z^2, the set of all pairs of integers, can be visualized as the set of names of the spatial locations of individual machines. Any mapping c: $Z^2 \to \Sigma$ is called a configuration. Let C denote the set of all configurations. We designate one symbol in Σ representing the quiescent state. A configuration c is called finite if $c(i)$ is quiescent for all but finitely many cells $i \in Z^2$. Let $\bar{C}$ denote the set of all finite configurations. Let $X = (\mathfrak{x}_1 ,..., \mathfrak{x}_9)$ be the Moore's neighborhood index. That is, $\{\mathfrak{x}_1 ,..., \mathfrak{x}_9\} = \{\mathfrak{x}_i = (x_i^{(1)}, x_i^{(2)}) \mid -1 \leqslant x_i^{(1)} \leqslant 1, -1 \leqslant x_i^{(2)} \leqslant 1, \mathfrak{x}_i \in Z^2\}$. The neighborhood index X and a local map: $\Sigma^9 \to \Sigma$ define a global map τ: $C \to C$ as usual. A configuration in the wider sense is a mapping: $D \to \Sigma$, where $D \subseteq Z^2$. $f_1: D_1 \to \Sigma$ and $f_2: D_2 \to \Sigma$ are equivalent if and only if there exists $\mathfrak{k} \in Z^2$ such that[1] $D_1 + \{\mathfrak{k}\} = D_2$ and $f_1(\mathfrak{i}) = f_2(\mathfrak{i} + \mathfrak{k})$ for all $\mathfrak{i} \in D_1$.[2] Let $\mathscr{C}$ denote the set of all the configurations in wider sense. Note that $C \subseteq \mathscr{C}$. The equivalence classes on $\mathscr{C}$ determined by the above defined equivalence relation is called patterns. $[f]$ denotes the pattern containing the configuration f in the wider sense. A global map $\tau: C \to C$ is easily generalized to be a map: $\mathscr{C} \to \mathscr{C}$. That is, for $f_1: D_1 \to \Sigma$ and $f_2: D_2 \to \Sigma$,

$$f_1\tau = f_2 \Leftrightarrow \begin{cases} \text{(i)} & D_2 + \{\mathfrak{x}_1, \ldots, \mathfrak{x}_9\} = D_1 \text{ and there exist } c_1 \text{ and } c_2 \\ & \text{such that } c_1\tau = c_2 \text{ and } c_1 \text{ and } c_2 \text{ are extensions}^2 \\ & \text{of } f_1 \text{ and } f_2 \text{, respectively, or} \\ \text{(ii)} & \text{for any } D \subseteq Z^2,\ D + \{\mathfrak{x}_1, \ldots, \mathfrak{x}_9\} \neq D_1 \text{ and } D_2 = \varnothing. \end{cases}$$

Let $\mathscr{P} = \{[f] \mid f \in \mathscr{C}\}$. From $\tau: C \to C$, $\tau: \mathscr{P} \to \mathscr{P}$ is induced as follows.

$$[f_1]\tau = [f_2] \Leftrightarrow \text{There exists } f_1 \in [f_1] \text{ and } f_2 \in [f_2] \text{ such that } f_1\tau = f_2.$$

DEFINITION. τ is k-balanced if and only if

$$|\{p \in \mathscr{P} \mid p\tau = p_0\}| = q^{(k+2)^2}/q^{k^2}$$

for any pattern p_0 of size $k \times k$, where $|S|$ denotes the number of elements in S. τ is balanced if and only if τ is k-balanced for any $k \geqslant 1$.

THEOREM. *$\tau: \bar{C} \to \bar{C}$ is one to one if and only if τ is balanced.*

Proof. Assume that τ is not balanced. Then, there exists a pattern p_0 of size $k \times k$ such that

$$|\{p \in \mathscr{P} \mid \tau(p) = p_0\}| > q^{(k+2)^2}/q^{k^2} = q^{4k+4}.$$

Let $j = |\{p \in \mathscr{P} \mid \tau(p) = p_0\}|$. We consider, for an integer i to be chosen later in the proof, patterns of size $i(k+2) \times i(k+2)$, as shown in Fig. 1, where each center of the i^2 patterns of size $(k+2) \times (k+2)$ is p_0. There are $q^{(ik+2i-2)^2 - i^2k^2}$ distinct patterns of size $(ik + 2i - 2) \times (ik + 2i - 2)$ which are constructed by removing a border of width 1 from the pattern of size $i(k+2) \times i(k+2)$. Then, there are j^{i^2} patterns of size $i(k+2) \times i(k+2)$ which are mapped by τ to one of the patterns of size $(ik + 2i - 2) \times (ik + 2i - 2)$ stated above. Clearly at least there are $j^{i^2}/q^{8\times(ik+2i-4)+16}$

[1] For $D_1, D_2 \subseteq Z^2$, $D_1 + D_2 = \{\mathfrak{d}_1 + \mathfrak{d}_2 \mid \mathfrak{d}_1 \in D_1, d_2 \in D_2\}$.

[2] $c: Z^2 \to \Sigma$ is a extension of $f: D \to \Sigma$ if and only if $f(\mathfrak{d}) = c(\mathfrak{d})$ for any $\mathfrak{d} \in D$.

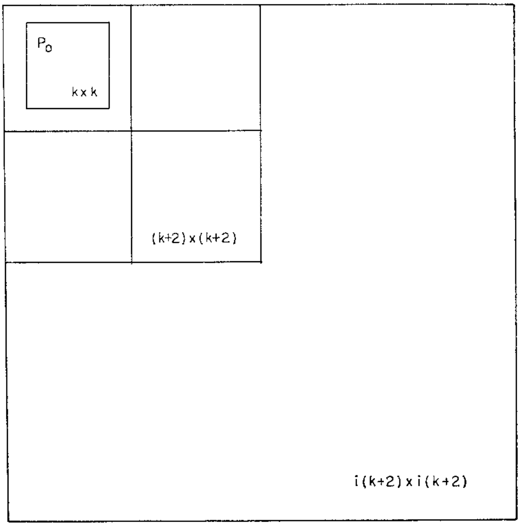

FIG. 1. An array of size $i(k+2) \times i(k+2)$, for use in proving the theorem.

patterns among the j^{i^2} patterns of size $i(k+2) \times i(k+2)$ that have a same border of width 2. On the other hand, since $j > q^{4k+4}$, there is a positive integer i such that

$$j^{i^2}/q^{8\times(ik+2i-4)+16} > q^{(ik+2i-2)^2-i^2k^2}.$$

Therefore, there are at least two patterns of size $i(k+2) \times i(k+2)$ with same width 2 border that are mapped into a same pattern of size $(ik+2i-2) \times (ik+2i-2)$. Thus there are two finite configurations c_1, c_2 ($c_1 \neq c_2$) such that $c_1\tau = c_2\tau$.

Conversely, assume $\tau: \bar{C} \to \bar{C}$ is not one to one. Then there are mutually erasable patterns (Moore, 1962). Thus, since there is a Garden of Eden configuration by Moore's theorem, τ is not balanced. Q.E.D.

If τ is not balanced, $\tau: \bar{C} \to \bar{C}$ is not one to one by Theorem 1. Therefore, by Moore's theorem, there is a Garden of Eden configuration. That is, for some k there exists a pattern p_0 of size $k \times k$ such that

$$\{p \in \mathscr{P} \mid \tau(p) = p_0\} = \varnothing.$$

Thus, we have the following theorem.

THEOREM 2. *τ is balanced if and only if for any k and for any pattern p_0 of size $k \times k$,*

$$\{p \in \mathscr{P} \mid \tau(p) = p_0\} \neq \varnothing.$$

Let $f_k\colon D_k \to \Sigma$, where $D_k = \{\mathfrak{x} = (x^{(1)}, x^{(2)}) \mid -k \leqslant x^{(i)} \leqslant k, i = 1, 2\}$. Let $0 \leqslant m < k$ and let $D_{k-m} = \{\mathfrak{x} = (x^{(1)}, x^{(2)}) \mid -(k-m) \leqslant x^{(i)} \leqslant k - m, i = 1, 2\}$. Let us denote by f_{k-m} the restriction of f_k to D_{k-m}. Then the center of $[f_k]$ with depth m is defined to be $[f_{k-m}]$.

DEFINITION 2. τ is hard if and only if there exists an integer $m > 0$ such that for any $k > 2m$ and for any pattern p_0 of size $k \times k$, when $\{p \in \mathscr{P} \mid \tau(p) = p_0\}$ is not empty, the center of the patterns in $\{p \in \mathscr{P} \mid \tau(p) = p_0\}$ with depth m is unique.

LEMMA 3 (Richardson, 1972). *$\tau\colon C \to C$ is one to one if and only if $\tau^{-1}\colon C \to C$ is a global map defined by some local map, where τ^{-1} means the inverse map of τ.*

THEOREM 3. *$\tau\colon C \to C$ is one to one if and only if τ is hard.*

Proof. Let $\tau\colon C \to C$ be one to one. Then from Lemma 3 τ is hard. Conversely, let $\tau\colon C \to C$ be not one to one. Then there exist c, c_1, and c_2 $(c_1 \neq c_2)$ such that $\tau(c_1) = \tau(c_2) = c$. Without loss of generality, let $c_1(\mathfrak{o}) \neq c_2(\mathfrak{o})$, where $\mathfrak{o} = (0, 0) \in Z^2$. Let $D_j = \{\mathfrak{x} = (x^{(1)}, x^{(2)}) \mid -j \leqslant x^{(i)} \leqslant j, i = 1, 2\}$. And let f_j be the restriction of c to D_j. Then for any j and any m with $j > m > 0$, there are at least two centers of the patterns in $\{p \in \mathscr{P} \mid \tau(p) = [f_j]\}$ with depth m. Q.E.D.

Since $\tau\colon C \to C$ is one to one implies that $\tau\colon \bar{C} \to \bar{C}$ is one to one (Richardson, 1972), the next theorem follows.

THEOREM 4. *If τ is hard, then τ is balanced.*

In view of Theorem 4, we obtain Definition 2′, which is equivalent to Definition 2.

DEFINITION 2′. τ is hard if and only if there exists an integer $m > 0$ such that for any $k > 2m$ and for any pattern p_0 of size $k \times k$, the center of the patterns in $\{p \in \mathscr{P} \mid \tau(p) = p_0\}$ with depth m is unique.

A configuration p is L.G.O.E. (Local Garden of Eden) if p is a pattern of size $k \times k$ such that $p \notin (\mathscr{P})$ (Aggarwal, 1973). The next diagram shows the relation between the results obtained in this paper and those obtained by Richardson (1972), Moore (1962), and Myhill (1963),

$$\begin{tikzcd}
 & & \text{There exists no L.G.O.E.} \arrow[d, leftrightarrow] & \\
 & & \begin{array}{l}\text{There exists no mutually}\\ \text{erasable configuration}\end{array} \arrow[d, leftrightarrow] & \\
\tau \text{ is hard} \arrow[d, leftrightarrow] & & \begin{array}{l}\text{There exists no indistinguishable}\\ \text{configuration}\end{array} \arrow[d, leftrightarrow] & \\
\tau : C \to C \text{ is one to one} & & \tau : \bar{C} \to \bar{C} \text{ is one to one} \arrow[r, leftrightarrow] & \tau \text{ is balanced} \\
 & \tau : C \to C \text{ is onto} \arrow[ur, leftrightarrow] & \tau : \bar{C} \to \bar{C} \text{ is onto} \arrow[u] &
\end{tikzcd}$$

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